# Supplementary Materials for "A Tunable Coupler for Superconducting Microwave Resonators Using A Nonlinear Kinetic Inductance Transmission Line"

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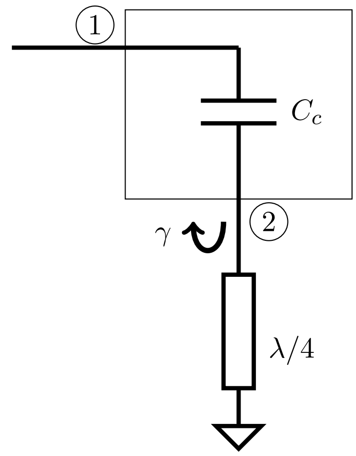


FIG. S1. Network model of a 1-port resonator capacitively coupled to a feedline.

## I. NETWORK ANALYSIS OF 1-PORT RESONATOR WITH FIXED COUPLING CAPACITOR AND THE DERIVATION OF $Q_c$

In this note, we first consider the network of a quarterwave resonator constructed from a shorted transmission line capacitively coupled to a feedline in a 1-port reflection measurement configuration, as shown in Fig. S1. The reflection coefficient $\gamma$ of the resonator is

$$\gamma = -e^{-2i\beta l(1-i\frac{1}{2Q_i})}$$

where $\beta$ is the propagation constant of the resonator transmission line, $l$ is the length, and $Q_i$ is the internal quality factor. Near the quarterwave resonance frequency $f \approx f_{\lambda/4} = v_p/4l$ so that the approximation

$$\beta l \approx \frac{\pi f}{2 f_{\lambda/4}},$$

can be made and

$$\frac{1}{\gamma} \approx 1 + \frac{\pi}{2Q_i} + i\pi \frac{f - f_{\lambda/4}}{f_{\lambda/4}}.$$

The reflection coefficient $R_{11}$ from the capacitor/resonator network is

$$R_{11} = S_{11} + \frac{S_{12}\gamma S_{21}}{1 - S_{22}\gamma}$$

where the scattering matrix for the capacitor is

$$\mathbf{S} = \begin{bmatrix} 1 - i\,2\delta_0 & i\,2\sqrt{\delta_0\delta_r} \\ i\,2\sqrt{\delta_0\delta_r} & 1 - 2\delta_0\delta_r - i\,2\delta_r \end{bmatrix} \tag{S1}$$

with $\delta_0 = \omega C_c Z_0$ and $\delta_r = \omega C_c Z_r$. $Z_0$ and $Z_r$ are the line impedances of the feedline and the quarterwave resonator.

Define the coupling quality factor $Q_c$ as

$$Q_c = 2\pi \frac{\text{energy stored in resonator}}{\text{energy leak from port 2 to port 1 per cycle}} = \frac{2\pi}{2|S_{21}|^2} = \frac{\pi}{4\delta_0\delta_r}. \tag{S2}$$

The factor of 2 in the denominator arises from the wave reflecting from the port two times per cycle for a quarterwave resonator when $f = f_{\lambda/4}$.

Because the coupler network is lossless, $|S_{21}|^2 + |S_{22}|^2 = 1$ and $S_{22}$ can be expressed as

$$S_{22} = \sqrt{1 - |S_{21}|^2} e^{i\phi} \approx 1 - \frac{\pi}{2Q_c} + i\phi,$$

where $\phi \ll 1$ is a small phase. Neglecting higher order terms, $R_{11}$ can be re-written as

$$R_{11} = 1 + \frac{S_{21}{}^2}{\frac{1}{\gamma} - S_{22}} = 1 - \frac{\frac{\pi}{Q_c}}{\frac{\pi}{2Q_i} + \frac{\pi}{2Q_c} + i\pi\left(\frac{f - f_{\lambda/4}}{f_{\lambda/4}} - \frac{\phi}{\pi}\right)}.$$

Defining the resonance frequency $f_r$ and quality factor $Q$ as

$$f_r = f_{\lambda/4}\left(1 + \frac{\phi}{\pi}\right), \; 1/Q = 1/Q_i + 1/Q_c,$$

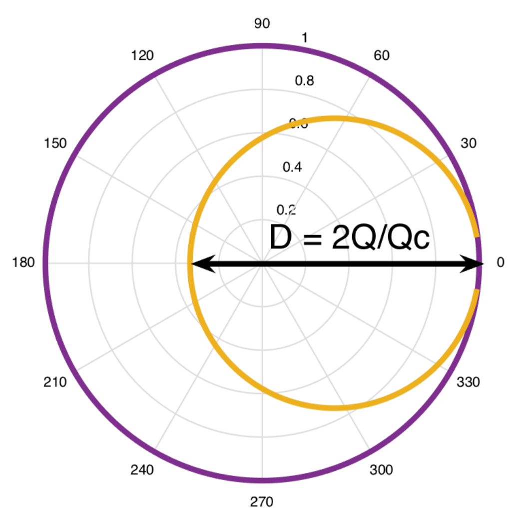


FIG. S2. The diameter of a 1-port resonance loop is $D = 2Q/Q_c$. If the resonator is overcoupled $D < 1$, if it's under coupled $D > 1$, and critical coupling yields $D = 1$.

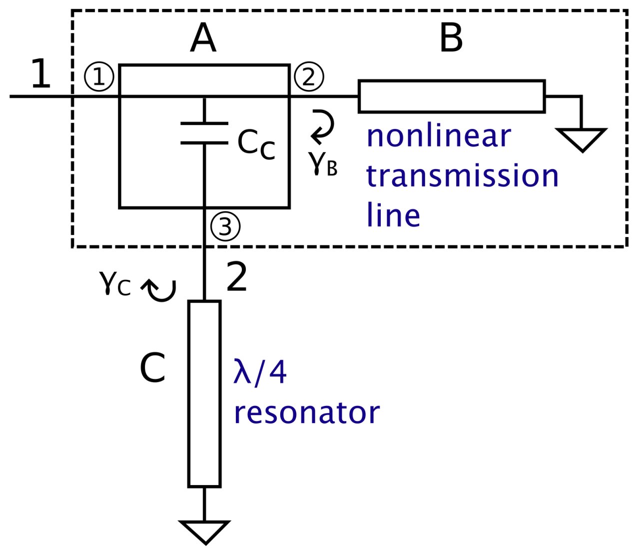


FIG. S3. Network model of the tunable coupler scheme. Reflection measurements are made at port 1.

we simplify $R_{11}$ and obtain

$$R_{11} \approx 1 - \frac{2\,Q/Q_c}{1 + i\,2\,Q\left(\frac{f-f_r}{f_r}\right)}. \tag{S3}$$

When $f = f_r$, $R_{11} = 1 - 2Q/Q_c$ and the diameter of the resonance circle is $D = 2Q/Q_c$ as shown in Fig. S2. The condition for critical coupling is $Q_i = Q_c$, implying $D = 1$ and $R_{11} = 0$ when $f = f_r$. This means that 100% of the power is dissipated in the resonator when it is driven at its resonance frequency under critical coupling conditions.

## II. NETWORK ANALYSIS OF TUNABLE COUPLER WITH RESONATOR AND THE DERIVATION OF EFFECTIVE COUPLING $Q_c'$

The tunable coupler scheme presented in this paper consists of a resonator capacitively coupled to a feedline with 1-port of the feedline connected to a shorted nonlinear transmission line as shown in Fig. S3. The whole network consists of 3 blocks, the capacitor coupler (A), the shorted nonlinear transmission line (B), and the quarterwave resonator (C). The combination of blocks A and B forms a new 2-port coupling block indicated by the dashed rectangle and our goal is to derive the coupling between port 1 and port 2.

The normalized scattering matrix for network block $A$ is given in Ref. (1),

$$\mathbf{S}^A = \begin{bmatrix} -i\,\delta_0/2 & 1 - i\delta_0/2 & i\sqrt{\delta_0\delta_r} \\ 1 - i\delta_0/2 & -i\delta_0/2 & i\sqrt{\delta_0\delta_r} \\ i\sqrt{\delta_0\delta_r} & i\sqrt{\delta_0\delta_r} & 1 - i\,2\delta_r \end{bmatrix}$$

where $\delta_0 = \omega C_c Z_0$, $\delta_r = \omega C_c Z_r$, $Z_0$ and $Z_r$ are the line impedances of the feedline and the quarterwave resonator.

The scattering parameters of the $A + B$ 2-port network are

$$\begin{aligned} S_{11}^{AB} &= S_{11}^{A} + \frac{S_{12}^{A}\,\gamma_B\,S_{21}^{A}}{1 - S_{22}^{A}\,\gamma_B} \approx \gamma_B \\ S_{21}^{AB} &= S_{31}^{A} + \frac{S_{21}^{A}\,\gamma_B\,S_{32}^{A}}{1 - S_{22}^{A}\,\gamma_B} \approx S_{31}^{A}(1+\gamma_B) = S_{12}^{AB} \\ S_{22}^{AB} &= S_{33}^{A} + \frac{S_{23}^{A}\,\gamma_B\,S_{32}^{A}}{1 - S_{22}^{A}\,\gamma_B} \approx \sqrt{1 - |S_{21}^{AB}|^2} e^{i\phi}, \end{aligned} \tag{S4}$$

where higher order terms have been neglected.

Finally, we include the resonator in the network model to write down the reflection coefficient from the entire network (blocks A + B + C),

$$R_{11}^{ABC} = S_{11}^{AB} + \frac{S_{12}^{AB}\,\gamma_C\,S_{21}^{AB}}{1 - S_{22}^{AB}\,\gamma_C},$$

where $\gamma_C$ is the reflection coefficient from the resonator. $R_{11}^{ABC}$ is the quantity we measure.

Plugging in the various quantities and following a derivation similar to the previous section, we find

$$R_{11}^{ABC} = \gamma_B - \frac{\delta_0\delta_r(1+\gamma_B)^2}{\left(\frac{\delta_0\delta_r(1+\gamma_B)^2}{2} + \frac{\pi}{2Q_i}\right) + i\,\pi\left(\frac{f - f_{\lambda/4}}{f_{\lambda/4}} - \frac{\phi}{\pi}\right)}.$$

After defining

$$f_r = f_{\lambda/4}\left(1 + \frac{\phi}{\pi}\right),\; Q_c' = 4Q_c\frac{\gamma_B}{(1+\gamma_B)^2},\; 1/Q = 1/Q_i + 1/Q_c', \tag{S5}$$

$R_{11}^{ABC}$ can be reduced to

$$R_{11}^{ABC} = \gamma_B\left(1 - \frac{2Q/Q_c'}{1 + i\,2Q\left(\frac{f - f_r}{f_r}\right)}\right). \tag{S6}$$

The terms in parenthesis have the same form as Eqn. (S3), implying that the standard fitting procedure for 1-port resonators will be valid for the tunable coupler.

Because the reflection coefficient from a lossless transmission line shorted on one end has the property $|\gamma_B| = 1$, it is possible to rewrite (S5) as

$$Q'_c = \frac{Q_c}{\cos^2\left(\frac{\angle\gamma_B}{2}\right)} \tag{S7}$$

where $\angle\gamma_B$ is the phase of $\gamma_B$.

## III. COMPARISON OF TRANSMISSION COEFFICIENT $t$ AND COUPLING QUALITY FACTOR $Q_c$ FOR THE FIXED AND TUNABLE COUPLER CASES

A 1-port capacitively coupled resonator is analog to a one-sided optical cavity with a partially transmissive wall as shown in Fig. 1 (a) in the main paper. According to Eqn. (S1) and Eqn. (S4), the transmission coefficient $t$ is

$$t = S_{21} = 2i\omega C_c\sqrt{Z_0 Z_r} \tag{S8}$$

for the fixed coupler case (Fig. S1) and

$$t' = S_{21}^{AB} = i\omega C_c\sqrt{Z_0 Z_r}(1+\gamma_B) \tag{S9}$$

for the tunable coupler case (Fig. S3). It follows from Eqn. (S5), (S7), (S8) and (S9) that $t'$ and $Q'_c$ are related to $t$ and $Q_c$ by

$$t' = \frac{t}{2}(1+\gamma_B)$$
$$Q'_c = \frac{4Q_c}{|1+\gamma_B|^2}$$

From both the signal graph in Fig. S3 and the expression $S_{21}^{AB}$ we see that the factor $1+\gamma_B$ arises from the interference between the incident wave and the reflected wave from the nonlinear transmission line.

## IV. ADDITIONAL PHASE FROM HIGHER ORDER NONLINEARITY AND CHANGING IMPEDANCE

$I_{\mathrm{dc}}$ affects the phase shift of the nonlinear transmission line in two ways. First, $I_{\mathrm{dc}}$ changes the phase velocity $v_p = 2\pi f/\beta$ as shown in Eqn. (4) of the main article. Second, $I_{\mathrm{dc}}$ slightly alters the characteristic impedance of the line, which we will discuss here and show that this effect is small compared to that of the changing phase velocity.

The reflection coefficient of the shorted nonlinear transmission line is

$$\gamma_B = \frac{i\,z\,\tan\beta l - 1}{i\,z\,\tan\beta l + 1},$$

where $z = Z_{\mathrm{NTL}}/Z_0 = 1 + \frac{\alpha}{2}\left(\frac{I}{I_*}\right)^2$ is the ratio of the characteristic impedance of the line to the impedance of the measurement system.

We define $\delta = \frac{\alpha}{2}\left(\frac{I}{I_*}\right)^2$. The phase of $\gamma_B$ can be written as

$$\angle\gamma_B = -2\arctan\left[(1+\delta)\tan\left[\beta l\right]\right] + \pi. \tag{S10}$$

Expanding $\angle\gamma_B$ to first order in $\delta$ and neglecting higher order terms,

$$\angle\gamma_B \approx \pi - 2\beta l - 2\delta\frac{\tan\beta l}{1+\tan^2\beta l}. \tag{S11}$$

An upper bound on the effect of the $\delta$ term is set by

$$\left|\frac{\tan\beta l}{1+\tan^2\beta l}\right| \le \frac{1}{2},$$

so any additional phase shift from changing impedance is no greater than $\delta$.

From time domain reflectometry, we measured a 4.5 ns round-trip electrical delay of the fishbone line. We estimate $2\beta_0 l \approx 126$ radians at 4.548 GHz (the resonance frequency of the resonator). Meanwhile, the current-induced phase shift had a maximum value of $\approx 4$ radians (Fig. 5(a)), implying a maximum kinetic inductance nonlinearity in Eqn. (4) of $\delta_{\max} = 4/126 = 0.032 \ll 1$. Therefore we conclude that the effect on the phase shift from changing impedance is no greater than 0.032 rad, which is much smaller than the effect from the changing $v_p$.

As discussed in the main paper and Ref. (2), a quartic term is non-negligible in the nonlinear current-dependence of the kinetic inductance. In Fig. S4 we show a fit that includes an $I_{\mathrm{dc}}^4$ term. The model fits the data well, suggesting that the quartic term may explain the departure of the $\theta(I_{\mathrm{dc}}^2)$ curve from a straight line in Fig. 5(a) in the main paper.

## REFERENCES

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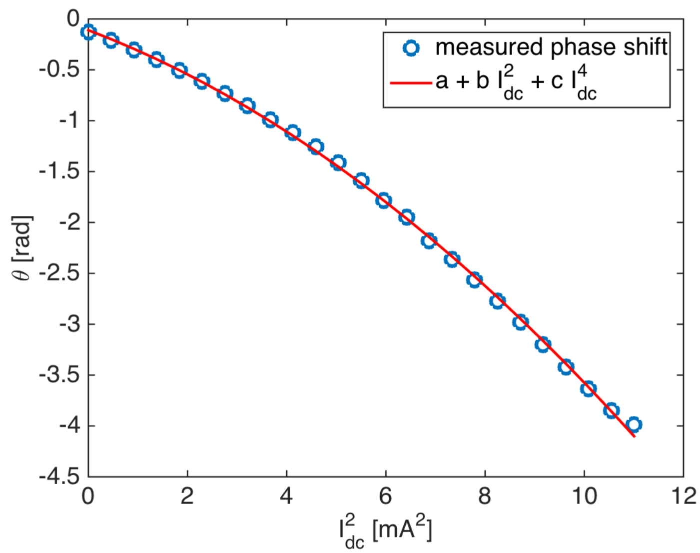


FIG. S4. Phase vs. $I^2_{\text{dc}}$ and quartic polynomial fit. The fit parameters were $b$ = -0.185 mA$^{-2}$ and $c$ = -0.016 mA$^{-4}$.